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A Note on Birth-Death-Immigration process in presence of Single or Twin Birth

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ABSTRACT
The present paper is a generalization of the linear growth process by considering the probabilities that a birth is a single birth or a twin birth. Explicit expression for the probabilities $P_n(t)$ was derived and expressions for mean and variance was obtained from the probability generating function of the process. Also, we have obtained the expression for mean and variance directly from the differential difference equation of the process. A simulation study was done to analyze the effects of birth in presence of single as well as twin births and deaths.

KEYWORDS
Birth, Death, Immigration Process, Linear Growth Process, Single or Twin Birth

AMS Subject Classification: 62M99, 60J80, 60J27, 60G07

1. Introduction

The population size in a specified geographical territory change with respect to time by the Births, Deaths as well as Immigration and Emigration. There are several literatures are available on different generalized Poisson Processes such as Birth-Immigration Process, Death-Immigration process, Birth-Death-Immigration Process, Birth-Death-Immigration-Emigration process etc., one can refer literature like Bailey (1964), Feller (1959), Gani and Stals (2007), Pradhan et. al. (2021). However, Dash and Pradhan (2021) considered a Birth process in presence of a single or a twin birth. An attempt has been made here to understand a Birth-Death-Immigration process in presence of Single or Twin Birth. Here we have derived probability generation function of this process. The expected value and the variance of the number of individuals present is derived from this probability generating function and again, the results are validated by direct derivation. We have also simulated the system by taking some specific value of the parameters.

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2. Assumption.

a. If n individuals are present at the instant from which the interval commences, the probability of one birth will occur in any short interval of length h is $n\lambda h + o(h)$ and the probability of more than one birth occurring in that small interval is $o(h)$. Birth occurring in $(t, t+h)$ are independent of time since the last occurrence.

b. If n individuals are present at the instant from which the interval commences, the probability of one death will occur in any short interval of length h is $n\mu h + o(h)$ and the probability of more than one death occurring in that small interval is $o(h)$. Death occurring in $(t, t+h)$ are independent of time since the last occurrence.
c. During the small interval of time $(t, t+h)$, the probability that a new member being added to the population by immigration is $\nu h + o(h)$ and the probability that more than one individual is added to the population in that small interval of length h is $o(h)$. Immigration occurring in $(t, t+h)$ are independent of time since the last occurrence.
d. For the same population, there is no interaction among the birth and death in small interval $(t, t+h)$ of time.

3. Probability Generating Function

Let $p_n(t)$ be the probability that the process starts with n individuals at instant t i.e.,

$$p_n(t) = Pr[N(t) = n].$$

Let us assume that a single birth occurs with probability $'p'$ and a twin birth occurs with probability $'q'$. Such that $p + q = 1$. To calculate $p_n(t+h)$ at the next time point $(t+h)$, the system can be instate E_n only if one of the following conditions are satisfied:

a. At time t, the population consist of n individuals and no birth, no death and no immigration occur during the time interval $(t, t+h)$.
b. At time t, the population consist of $(n-1)$ individuals and single birth occurs during the next time interval $(t, t+h)$ with probability $'p'$, but no death and no immigration occur during that interval of time.
c. At time t, the population consist of $(n-2)$ individuals and a twin birth occurs during the next time interval $(t, t+h)$ with probability $'q'$ but no death and no immigration occur during same interval of time.
d. At time t, the population consist of $(n+1)$ individuals and a death occurs during the next time interval $(t, t+h)$ but no birth and no immigration occur during that interval of time.
e At time t, the population consist of $(n-1)$ individuals and a new member is added to the population by immigration during the time interval $(t, t+h)$ but no birth, no death occur during that interval of time.
f. During $(t, t+h)$ two or more transitions occur with probability $o(h)$ i.e., the occurrence of more than one transition in $(t, t+h)$ is negligible.

All the above contingencies are mutually exclusive and hence for $n \geq 1$, we have

We have,

$$\begin{aligned}
p_n(t+h) &= p_n(t)\left[1-n\lambda h+o(h)\right]\left[1-n\mu h+o(h)\right]\left[1-\nu h+o(h)\right] \\
&\quad +p_{n-1}(t)\left[\overline{n-1}\lambda ph+o(h)\right]\left[1-\overline{n-1}\mu h+o(h)\right]\left[1-\nu h+o(h)\right] \\
&\quad +p_{n-2}(t)\left[\overline{n-1}\lambda qh+o(h)\right]\left[1-\overline{n-2}\mu h+o(h)\right]\left[1-\nu h+o(h)\right] \\
&\quad +p_{n-1}(t)\left[1-\overline{n-1}\lambda h+o(h)\right]\left[1-\overline{n-1}\mu h+o(h)\right]\left[\nu h+o(h)\right] \\
&\quad +p_{n+1}(t)\left[1-\overline{n+1}\lambda h+o(h)\right]\left[\overline{n+1}\mu h+o(h)\right]\left[1-\nu h+o(h)\right] \\
&\quad +o(h), \quad \text{for } \lambda \neq \mu. \qquad (1) \\
\Rightarrow p_n(t+h) &= p_n(t)-n\lambda hp_n(t)-n\mu hp_n(t)-\nu hp_n(t)-(n-1)\lambda php_{n-1}(t) \\
&\quad +(n-2)\lambda qhp_{n-2}(t)+\nu hp_{n-1}(t)+(n+1)\mu hp_{n+1}(t)+o(h) \qquad (2)
\end{aligned}$$

Now,dividing h and taking $h \to 0$ in both sides, we get

$$\begin{aligned}
\lim_{h\to 0}\frac{p_n(t+h)-p_n(t)}{h} &= p_n(t)-n\lambda p_n(t)-n\mu p_n(t)-\nu p_n(t) \\
&\quad -(n-1)\lambda pp_{n-1}(t)+(n-2)\lambda qp_{n-2}(t)+\nu p_{n-1}(t) \\
&\quad +(n+1)\mu p_{n+1}(t)+\lim_{h\to 0}\frac{o(h)}{h} \qquad (3) \\
\Rightarrow p_n'(t) &= -n\lambda p_n(t)-n\mu p_n(t)-\nu p_n(t) \\
&\quad -(n-1)\lambda pp_{n-1}(t)+(n-2)\lambda qp_{n-2}(t) \\
&\quad +\nu p_{n-1}(t)+(n+1)\mu p_{n+1}(t) \quad \text{for } n \geq 1. \qquad (4)
\end{aligned}$$

For $n = 0$, we can have

$$p_0'(t) = -\nu p_o(t) + \mu p_1(t) \qquad (5)$$

Now,combining the equations (4) and (5), the differential-difference equation of the process is given by

$$\begin{aligned}
p_n'(t) &= -n\lambda p_n(t)-n\mu p_n(t)-\nu p_n(t)-(n-1)\lambda pp_{n-1}(t) \\
&\quad +(n-2)\lambda qp_{n-2}(t)+\nu p_{n-1}(t)+(n+1)\mu p_{n+1}(t) \quad \text{for } n \geq 0. \qquad (6)
\end{aligned}$$

The generating function of the process is given by

$$\pi(z,t) = \sum_{n=0}^{\infty} p_n(t)z^n, \quad |z| \leq 1 \qquad (7)$$

Using the equation (7), we have

$$\frac{d\log\pi(z,t)}{1} = -\frac{\nu dz}{[\lambda z(qz+1)-\mu]} = \nu(z-1)dt \qquad (8)$$

If the process starts with n_0 individuals at instant time $t = 0$, then $p_{n_0}(0) = 1$ and

$p_k(0) = 0$ for $k \neq n_0$. Hence, $\pi(z, 0) = z^{n_0}$. Thus, from equation (8),

$$\begin{aligned}
&\frac{z^{n_0}}{\left[\dfrac{(\lambda q z^2+\lambda z-\mu)^{\frac{1}{2}}\left|(z+\frac{1}{q})-\sqrt{(\frac{\mu}{\lambda q}+\frac{1}{4q^2})}\right|^{\frac{\lambda(2q+1)}{4q\sqrt{(\frac{\mu}{\lambda q}+\frac{1}{4q^2})}}}}{(z-1)\left|(z+\frac{1}{q})+\sqrt{(\frac{\mu}{\lambda q}+\frac{1}{4q^2})}\right|^{\frac{\lambda(2q+1)}{4q\sqrt{(\frac{\mu}{\lambda q}+\frac{1}{4q^2})}}}}\right]^{\left(\frac{\nu}{[\lambda z(1+qz)-\mu]}\right)}} \\
&= \psi\left[\frac{\left(\lambda q z^2+\lambda z-\mu\right)^{\frac{1}{2}}\left|(z+\frac{1}{q})-\sqrt{(\frac{\mu}{\lambda q}+\frac{1}{4q^2})}\right|^{\frac{\lambda(2q+1)}{4q\sqrt{(\frac{\mu}{\lambda q}+\frac{1}{4q^2})}}}}{(z-1)\left|(z+\frac{1}{q})+\sqrt{(\frac{\mu}{\lambda q}+\frac{1}{4q^2})}\right|^{\frac{\lambda(2q+1)}{4q\sqrt{(\frac{\mu}{\lambda q}+\frac{1}{4q^2})}}}}\right]
\end{aligned} \tag{9}$$

Letting

$$u = \left[\frac{\left(\lambda q z^2+\lambda z-\mu\right)^{\frac{1}{2}}\left|(z+\frac{1}{q})-\sqrt{(\frac{\mu}{\lambda q}+\frac{1}{4q^2})}\right|^{\frac{\lambda(2q+1)}{4q\sqrt{(\frac{\mu}{\lambda q}+\frac{1}{4q^2})}}}}{(z-1)\left|(z+\frac{1}{q})+\sqrt{(\frac{\mu}{\lambda q}+\frac{1}{4q^2})}\right|^{\frac{\lambda(2q+1)}{4q\sqrt{(\frac{\mu}{\lambda q}+\frac{1}{4q^2})}}}}\right],$$

we can find z in terms of u and we can find the p.g.f. $\pi(z, t)$ of Birth-Death-Immigration process in presence of single or twin birth.

4. Mean and Variance of the Process

We have

$$\begin{aligned}
p_n'(t) &= -n\lambda p_n(t) - n\mu p_n(t) - \nu p_n(t) - (n-1)\lambda p p_{n-1}(t) \\
&\quad +(n-2)\lambda q p_{n-2}(t) + \nu p_{n-1}(t) + (n+1)\mu p_{n+1}(t) \quad \text{for } n \geq 1.
\end{aligned} \tag{10}$$

Multiplying both sides of (10) by n, we find

$$\begin{aligned}
M'(t) &= \sum_{n=0}^{\infty} n p_n'(t) \\
&= \sum_{n=0}^{\infty} n[-n\lambda p_n(t) - n\mu p_n(t) - \nu p_n(t) - (n-1)\lambda p p_{n-1}(t) \\
&\quad +(n-2)\lambda q p_{n-2}(t) + \nu p_{n-1}(t) + (n+1)\mu p_{n+1}(t)] \\
&= [\lambda(q+1) - \mu]\, M(t) + \nu && (11) \\
\Rightarrow \frac{dM(t)}{dt} &= [\lambda(q+1) - \mu]\, M(t) + \nu. && (12)
\end{aligned}$$

Integrating both sides, we get

$$M(t) = -\nu \frac{e^{-[\lambda(q+1)-\mu]t}}{[\lambda(q+1) - \mu]} + Ce^{[\lambda(q+1)-\mu]t} \tag{13}$$

where C is a constant to be determined by initial condition. If the process starts with n_0 individuals at time $t = 0$, We have $M(0) = n_0$.

$$C = n_0 + \frac{\nu}{[\lambda(q+1) - \mu]} \tag{14}$$

Using the value of C from the equation (14) in the equation (13), we get the value of $M(t)$ as

$$E[N(t)] = \frac{\nu}{[\lambda(q+1) - \mu]} \left[e^{[\lambda(q+1)-\mu]t} - 1\right] + n_0 e^{[\lambda(q+1)-\mu]t} \tag{15}$$

Multiplying both sides of (10) by n^2, we find

$$\begin{aligned} M_2'(t) &= \sum_{n=0}^{\infty} n^2 p_n'(t) \\ &= \sum_{n=0}^{\infty} n^2[-n\lambda p_n(t) - n\mu p_n(t) - \nu p_n(t) - (n-1)\lambda p p_{n-1}(t) \\ &\quad +(n-2)\lambda q p_{n-2}(t) + \nu p_{n-1}(t) + (n+1)\mu p_{n+1}(t)] \\ &= 2\left[\lambda(q+1) - \mu\right] M_2(t) + \left[\lambda(1+3q) + \mu + 2\nu\right] M(t) + \nu \end{aligned} \tag{16}$$

Integrating both sides we can get,

$$M_2(t) = -\left[\frac{\lambda(3q+1) + \mu + 2\nu}{\lambda(q+1) - \mu}\right] M(t) - \frac{\nu}{2\left[\lambda(q+1) - \mu\right]} + Ce^{2[\lambda(q+1)-\mu]t} \tag{17}$$

For $t = 0$, $M_2(0) = n_0^2$ and $M(0) = n_0$. So, using these results in equation (17), we get the value of C as

$$C = n_0^2 + \left[\frac{\lambda(3q+1) + \mu + 2\nu}{\lambda(q+1) - \mu}\right] n_0 + \frac{\nu}{2\left[\lambda(q+1) - \mu\right]} \tag{18}$$

Using the value of C from equation (18) in equation (17), we get

$$\begin{aligned} M_2(t) &= \left[\frac{\lambda(3q+1) + \mu + 2\nu}{\lambda(q+1) - \mu}\right] \left[n_0 e^{2[\lambda(q+1)-\mu]t} - M(t)\right] \\ &\quad + \frac{\nu}{2\left[\lambda(q+1) - \mu\right]} \left[e^{2[\lambda(q+1)-\mu]t} - 1\right] + n_0^2 e^{2[\lambda(q+1)-\mu]t} \end{aligned} \tag{19}$$

Hence, the variance of the process $N(t)$ is given by

$$\begin{aligned} V[N(t)] &= M_2(t) - [M(t)]^2 \\ &= \left[\frac{\lambda(3q+1)+\mu+2\nu}{\lambda(q+1)-\mu}\right]\left[n_0 e^{2[\lambda(q+1)-\mu]t} - M(t)\right] \\ &\quad + \frac{\nu}{2\left[\lambda(q+1)-\mu\right]}\left[e^{2[\lambda(q+1)-\mu]t} - 1\right] + n_0^2 e^{2[\lambda(q+1)-\mu]t} \\ &\quad - \left[\frac{\nu}{[\lambda(q+1)-\mu]}\left[e^{[\lambda(q+1)-\mu]t} - 1\right] + n_0 e^{[\lambda(q+1)-\mu]t}\right]^2 \end{aligned} \tag{20}$$

5. A Simulation Study

Here we performed simulation of Birth-Death-Immigration Process in presence of Twin or Single Birth and compared this process with Birth-Death-Immigration Process by taking different values of parameters.From equation(15), The expectation of Birth-Death-Immigration Process in presence of Twin or Single Birth is

$$E[N(t)] = M(t) = \frac{\nu}{[\lambda(q+1)-\mu]}\left[e^{[\lambda(q+1)-\mu]t} - 1\right] + n_0 e^{[\lambda(q+1)-\mu]t}.$$

The expectation of Birth-Death-Immigration Process is

$$E[N(t)] = M_1(t) = \frac{\nu}{\lambda-\mu}\left(e^{(\lambda-\mu)t} - 1\right) + n_0 e^{(\lambda-\mu)t}.$$

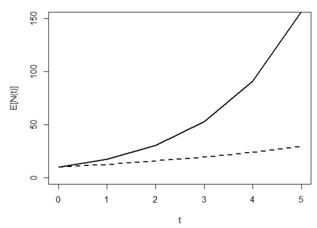

Figure 1. Mean of BDIP in presence of Twin or Single Birth ($\lambda > \mu$).

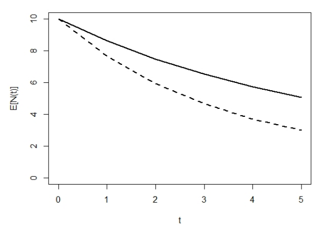

Figure 2. Mean of BDIP in presence of Twin or Single Birth ($\lambda < \mu$).

From Fig.(1) and Fig.(2),

$$\begin{aligned} M(t) &= \frac{\nu}{[\lambda(q+1)-\mu]}\left[e^{[\lambda(q+1)-\mu]t}-1\right]+n_0e^{[\lambda(q+1)-\mu]t} \\ &\geq \frac{\nu}{\lambda-\mu}\left(e^{(\lambda-\mu)t}-1\right)+n_0e^{(\lambda-\mu)t}=M_1(t) \end{aligned}$$

In case of $(\lambda > \mu)$ the expectation of Birth-Death-Immigration Process in presence of Twin or Single Birth is larger than the expectation of Birth-Death-Immigration Process. For large value of t in both the cases there is a chance of population explosion but in former case the growth rate will be high if $q > p$ and nearly equal if $q < p$.
In case of $(\lambda < \mu)$ the expectation of Birth-Death-Immigration Process in presence of Twin or Single Birth is larger than the expectation of Birth-Death-Immigration Process. For the large value of t both the process will extinct after certain transition if $q < p$. But it will show an opposite trend if $q > p$.

6. Conclusion

Birth-Death-Immigration Process in presence of Twin or Single Birth Can be converted to Birth-Death-Immigration Process for the certain value of p, i.e., $p = 1$ and a Twin Birth-Death-Immigration process for the certain value of q, i.e., $q = 1$. In absence of Immigration it can be Birth-Death process with single or twin birth.

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